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Feynman's Formula for a Harmonic Oscillator

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Feynman's original formula for the Green's function of a harmonic oscillator includes the Maslov correction. It also leads to the known formula for caustics for the time intervals that are equal to integer multiplets of the half period.

The well-known formula of Feynman for the harmonic oscillator Green's function reads (Feynman and Hibbs, 1965)

$$K(x_2, T; x_1, 0) = \left(\frac{m\omega}{2\pi i\hbar \sin \omega T}\right)^{1/2} \times \exp\left\{\frac{im\omega}{2\hbar \sin \omega T}\left[\left(x_1^2 + x_2^2\right)\cos \omega T - 2x_1x_2\right]\right\} \tag{1}$$

If the time interval T is given in terms of the period $\tau = 2\pi/\omega$ as

$$T = n\tau/2 + \delta, \qquad \text{with } n = 0, 1, 2, \ldots;\ 0 < \delta < \tau/2$$

we simply write

$$\begin{aligned} \sin \omega T &= e^{i\pi n} \sin \omega\delta \\ \cos \omega T &= e^{i\pi n} \cos \omega\delta \end{aligned} \tag{2}$$

then, Eq. (1) becomes

$$K(x_2, T; x_1, 0) = \left(\frac{m\omega}{2\pi\hbar \sin \omega\delta}\right)^{1/2} e^{-i(\pi/2)(1/2+n)} \times \exp\left\{\frac{im\omega}{2\hbar \sin \omega T}\left[\left(x_1^2 + x_2^2\right)\cos \omega T - 2x_1x_2\right]\right\} \tag{3}$$

which is the formula for the Green's function with Maslov correction observing $\sin\omega\delta = |\sin\omega T|$ (Horvathy, 1979).

For $\delta \to 0$ (i.e., at caustics), the above formula becomes, by substitution of (2) into (3) for $\sin\omega T$ and $\cos\omega T$,

$$K\left(x_2, \frac{n\tau}{2}; x_1, 0\right) = \lim_{\delta \to 0}\left(\frac{m}{2\pi\hbar\delta}\right)^{1/2} e^{-i(\pi/2)(1/2+n)} \times \exp\left\{\frac{im}{2\hbar\delta}\left[\left(x_1^2 + x_2^2\right) - e^{-i\pi n} 2x_1x_2\right]\right\} \tag{4}$$

By changing the parameter δ to a as $\delta = -ia^2, a > 0$ Eq. (4) takes the form

$$K\left(x_2, \eta\frac{\tau}{2}; x_1, 0\right) = \exp\left(-\frac{i\pi}{4}\right)\exp\left(-i\frac{\pi}{2}n\right)\left(\frac{m}{2\hbar}\right)^{1/2}\left(\frac{1}{-i}\right)^{1/2} \times\left\{\lim_{a\to 0}\frac{1}{\sqrt{\pi}\,a}\exp\left[-(m/2\hbar)\left(x_1 - (-)^n x_2\right)^2/a^2\right]\right\}$$

Since the limit in the above equation is a definition of the δ function (Harris, 1975), the kernel at caustics becomes

$$K\left(x_2, \frac{n2}{2}; x_1, 0\right) = \exp\left(-i\frac{\pi}{2}n\right)\left(\frac{m}{2\hbar}\right)^{1/2}\delta\left(\left(\frac{m}{2\hbar}\right)^{1/2}\left(x_1 - (-)^n x_2\right)\right)$$

or

$$K\left(x_2, \frac{n\tau}{2}; x_1, 0\right) = \exp\left(-i\frac{\pi}{2}n\right)\delta\left(x_1 - (-)^n x_2\right)$$

which is of the well-known form (Horvathy, 1979).

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